

Impredicative consistency and reflection

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Abstract

Given a set X of natural numbers, we may formalize “*The formula ϕ is a theorem of ω -logic over the theory T using an oracle for X* ” by an expression $[|X]_T\phi$, defined using a least fixed point in the language of second-order arithmetic. We will prove that the consistency and reflection principles arising from this notion of provability lead to axiomatizations of $\Pi_1^1\text{-CA}_0$ and $\Pi_1^1\text{-CA}_0$ with bar induction. We compare this to well-known results that reflection for ω -derivable formulas and ω -model reflection are equivalent to bar induction.

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1. Introduction

Reflection principles in formal arithmetic are statements of the form “*If ϕ is a theorem of T , then ϕ* ” [12]. Using notation from provability logic [4], for a computably enumerable theory T we may use $\Box_T\phi$ to denote a natural formalization of “ *ϕ is a theorem of T* ”. Then, the above statement may be written succinctly as $\Box_T\phi \rightarrow \phi$. If ϕ is a sentence, this gives us an instance of *local reflection*. Although such principles merely state the soundness of T , they can almost never be proven within T itself. For example, setting $\phi \equiv 0 \neq 1$, we see that $\Box_T\phi \rightarrow \phi$ is equivalent to $\sim\Box_T 0 \neq 1$, which asserts the consistency of T and hence is unprovable within T itself (if T satisfies the assumptions of Gödel’s second incompleteness theorem). More generally, by Löb’s theorem we have that $T \vdash \Box_T\phi \rightarrow \phi$ *only* if ϕ is already a theorem of T [13].

We can extend reflection to formulas $\phi(x)$, obtaining *uniform reflection principles*, denoted $\text{RFN}[T]$. These are given by the scheme

$$\forall x(\Box_T \phi(\bar{x}) \rightarrow \phi(x)),$$

where \bar{x} denotes the numeral of x .

Uniform reflection principles are particularly appealing because they sometimes give rise to familiar theories. If we use PRA to denote *primitive recursive arithmetic*, Kreisel and Lévy proved in [12] that

$$\text{PA} \equiv \text{PRA} + \text{RFN}[\text{PRA}];$$

in fact, we may replace PRA by the weaker *elementary arithmetic* (EA), obtained by restricting the induction schema in Peano arithmetic to Δ_0^0 formulas and adding an axiom asserting that the exponential function is total [2].

Recall that the ω -rule is an infinitary deduction rule that has the following form:

$$\frac{\phi(\bar{0}), \Gamma \quad \phi(\bar{1}), \Gamma \quad \phi(\bar{2}), \Gamma \quad \dots}{\forall x \phi(x), \Gamma},$$

and ω -logic is the logic generated by the ω -rule together with the standard finitary rules of the Tait calculus. More generally, ω -logic over T allows for sequents derivable in T to be used as axioms.

In this article, we will study formalizations of ω -reflection in second-order arithmetic; that is, statements of the form “If ϕ is a theorem of ω -logic, then ϕ ”. The question readily arises as to what it means for ϕ to be a theorem of ω -logic. There are at least three ways to model this. Informally, they are:

- (i) There is a well-founded derivation tree formalizing an ω -proof of ϕ , in which case we will write $[\mathbf{P}]\phi$.
- (ii) There is a well-order Λ such that ϕ belongs to the set of theorems of ω -logic defined by transfinite recursion on Λ , in which case we will write $[\mathbf{R}]\phi$.
- (iii) The formula ϕ belongs to the least set closed under the rules and axioms of ω -logic. If this is the case, we will write $[\mathbf{I}]\phi$.

Although we will discuss these in greater detail later, the ideas behind $[\mathbf{P}]\phi$ and $[\mathbf{I}]\phi$ should be clear; $[\mathbf{P}]\phi$ gives a ‘local’ view of ϕ being a theorem of ω -logic by considering (infinite) ω -proofs of ϕ , while $[\mathbf{I}]\phi$ gives a more global

perspective, describing the set of theorems of ω -logic as a whole via an inductive definition. Meanwhile, $[\mathbf{R}]\phi$ describes the approximations to the fixed point used in $[\mathbf{I}]\phi$ via transfinite recursion.

Over a strong enough formal theory, one can show that all of these notions of provability are equivalent. However, from the point of view of a weak theory, they may vary in strength. For $\mathbf{X} \in \{\mathbf{P}, \mathbf{R}, \mathbf{I}\}$ and $A \subseteq \mathbb{N}$, let us write $[\mathbf{X}|A]\phi$ if ϕ is provable in the sense of \mathbf{X} from the atomic diagram of A . Then, we define a schema

$$\omega_{\mathbf{X}}\text{-RFN} \equiv \forall A \forall n \left([\mathbf{X}|A]\phi(\bar{n}, \bar{A}) \rightarrow \phi(n, A) \right);$$

the notation \bar{A} indicates a second-order constant added to represent A . If Γ is a set of formulas, $\omega_{\mathbf{X}}\text{-RFN}_{\Gamma}$ is the restriction of this scheme to $\phi \in \Gamma$. Then, over RCA_0 we have that:

$$\omega_{\mathbf{P}}\text{-RFN} \equiv \Pi_{\omega}^1\text{-BI}_0; \quad (1)$$

$$\omega_{\mathbf{R}}\text{-RFN}_{\Pi_2^1} \equiv \text{ATR}_0. \quad (2)$$

(We will review the theories $\Pi_{\omega}^1\text{-BI}_0$ of full bar induction and ATR_0 of arithmetical transfinite recursion in §2). The first item is proven in [1] and the second in [6]. As we will see, if we use $\omega_{\mathbf{X}}\text{-RFN}_{\Gamma}[T]$ to denote a variant of the scheme where ω -logic is extended by theorems of T , (2) generalizes to

$$\omega_{\mathbf{R}}\text{-RFN}_{\Sigma_{n+1}^1}[\text{ACA}_0] \equiv \text{ATR}_0 + \Pi_n^1\text{-BI} \quad (3)$$

(which is just $\Pi_n^1\text{-BI}_0$ if $n > 1$). Moreover, (1) also holds for ω -model reflection, the scheme asserting that any formula true in every ω -model must be true [11]. This begs the question: is $\omega_1\text{-RFN}$ also equivalent to a natural theory? In this article, we answer the question affirmatively, and prove that:

$$\omega_1\text{-RFN}_{\Pi_3^1} \equiv \Pi_1^1\text{-CA}_0; \quad (4)$$

$$\omega_1\text{-RFN}_{\Sigma_{n+1}^1}[\text{ACA}_0] \equiv \Pi_1^1\text{-CA}_0 + \Pi_n^1\text{-BI}. \quad (5)$$

Both equivalences are proven over the theory ECA_0 of elementary comprehension, which is strictly weaker than RCA_0 or even RCA_0^* .

Layout of the article

In §2 we establish some basic notation we will use, and review the subsystems of second-order arithmetic that will be of interest to us. In §3 we review

formalizations of ω -logic in the literature, and in §4 we review ω -models, which give rise to another family of reflection principles, also equivalent to bar induction. In §5 we give our formalizations using inductive definitions. In §6 we discuss completeness results for ω -logic and prove (3), and §7 introduces the reflection principles based on our fixed point construction and proves partial results leading to (4) and (5). The latter are proven in §8 using β -models.

2. Second-order arithmetical theories

In this section we review some basic notions of second-order arithmetic and mention some important theories that will appear throughout the article.

2.1. Conventions of syntax

It will be convenient to work within a Tait-style calculus, so we will consider a language without negation, except on primitive predicates. Thus terms and formulas will be built from the symbols $0, 1, +, \times, \mathbf{exp}, =, \neq, \in, \notin$, representing the standard constants, operations and relations on the natural numbers, along with the Booleans \wedge, \vee and the quantifiers \forall, \exists . The *rank* of a formula is the number of logical symbols (Booleans and quantifiers) that appear in it. We assume a countably infinite set of first-order variables n, m, x, y, z, \dots , which will always be denoted by lower-case letters, as well as a countably infinite set of second-order variables. It will be convenient to assume that the second-order variables are enumerated by $\mathbf{V} = \langle V_i \rangle_{i \in \mathbb{N}}$, although we may also use X, Y, Z, \dots to denote set-variables. Tuples of first-order terms or second-order variables will be denoted with a boldface font, e.g. \mathbf{t}, \mathbf{X} . In general, if $\mathbf{S} = \langle S_i \rangle_{i \in \mathbb{N}}$ is a sequence we will write $\mathbf{S}_{<n}$ for $\langle S_i \rangle_{i < n}$. We also include countably many set-constants $\mathbf{O} = \langle O_i \rangle_{i \in \mathbb{N}}$, which will be used as ‘oracles’ (see §3.2).

We define $x \leq y$ by $\exists z (y = x + z)$ and $x < y$ by $x + 1 \leq y$. In the meta-language we may also use the symbol ‘=’, although sometimes we use ‘ \equiv ’ instead in order to distinguish it from the object-language equality. Since we have no negation in the language, we define $\sim\phi$ by using De Morgan’s laws and the classical dualities for quantifiers. In particular, we define $\phi \rightarrow \psi$ by $\sim\phi \vee \psi$. The set of all formulas will be denoted $\mathbf{\Pi}_\omega^1$.

Fix some elementary Gödel numbering mapping a formula $\psi \in \mathbf{\Pi}_\omega^1$ to a natural number $\ulcorner \psi \urcorner$; terms and sequents of formulas are also assigned Gödel numbers. Since we will be working mainly inside theories of arithmetic, we

will often identify ψ with $\ulcorner \psi \urcorner$. For a natural number n , define a term \bar{n} recursively by $\bar{0} = 0$ and $\overline{n+1} = (\bar{n}) + 1$. We will assume that the Gödel numbering has the natural property that $\ulcorner \psi \urcorner < \ulcorner \phi \urcorner$ whenever ψ is a proper subformula of ϕ .

We use Δ_0^0 to denote the set of all formulas, possibly with set parameters but without the occurrence of the set-constants O_i , where no second-order quantifiers appear and all first-order quantifiers are *bounded*, that is, of the form $\forall x < t \phi$ or $\exists x < t \phi$. Observe that in our presentation, a Δ_0^0 formula may contain exponential bounds. We simultaneously define $\Sigma_0^0 = \Pi_0^0 = \Delta_0^0$ and recursively define Σ_{n+1}^0 to be the set of all formulas of the form $\exists x \phi$ with $\phi \in \Pi_n^0$, and similarly Π_{n+1}^0 to be the set of all formulas of the form $\forall x \phi$ with $\phi \in \Sigma_n^0$. We denote by Π_ω^0 the union of all Π_n^0 ; these are the *arithmetical formulas*.

The classes Σ_n^1, Π_n^1 are defined analogously, but using second-order quantifiers, and setting $\Sigma_0^1 = \Pi_0^1 = \Delta_0^1 = \Pi_\omega^0$. It is well-known that every second-order formula is equivalent to another in one of the above forms. We use a lightface font for the analogous classes where no set-variables appear free: $\Delta_n^m, \Pi_n^m, \Sigma_n^m$. For lightface classes of formulas, we may write $\Gamma(\mathbf{Y})$ to indicate that the second-order variables in \mathbf{Y} may appear free (and no others). Finally, if Γ is a set of formulas and n is a natural number, we use Π_n^1/Γ to denote the set of formulas of the form $\forall X_n \exists X_{n-1}, \dots, Q_1 X_1 \phi$, with $\phi \in \Gamma$ and $Q_1 \in \{\forall, \exists\}$.

We will also use *pseudo-terms* to simplify notation, where an expression $\varphi(t(\mathbf{x}))$ should be understood as a shorthand for $\exists y (\psi(\mathbf{x}, y) \wedge \varphi(y))$, with ψ a Δ_0^0 formula defining the graph of the intended interpretation of t . Similarly, an *elementary pseudo-term* is an expression $\exists y < s(\mathbf{x}) (\psi(\mathbf{x}, y) \wedge \varphi(y))$, where s is a standard term bounding the values of $t(\mathbf{x})$. The domain of the functions defined by these pseudo-terms may be a proper subset of \mathbb{N} .

Let us list some of the (pseudo-)terms we will use:

1. An elementary term $\langle x, y \rangle$ which returns a code of the ordered pair formed by x and y and elementary projection terms so that $(\langle x, y \rangle)_0 = x$ and $(\langle x, y \rangle)_1 = y$. We will overload this notation by also using it for sequences, coded in a standard way. As with tuples of variables, we use a boldface font when a first-order object is meant to be regarded as a sequence. For a sequence \mathbf{s} , we will also use $(\mathbf{s})_i$ to denote an elementary pseudo-term which picks out the i^{th} element of \mathbf{s} if it exists, and is undefined otherwise, and $|\mathbf{s}|$ denotes an elementary pseudo-term

for the length of \mathbf{s} . If $n \in \mathbb{N}$, $\mathbf{s} \frown n$ denotes the sequence obtained by adjoining n to \mathbf{s} as its last element.

2. An elementary term \bar{x} mapping a natural number to the code of its numeral.
3. A (non-elementary) term $\llbracket x \rrbracket$ which, when x codes a closed term t , returns the value of t as a natural number.
4. For every formula ϕ and variables x_0, \dots, x_m , an elementary term $\phi(\dot{x}_0, \dots, \dot{x}_m)$ which, given natural numbers n_0, \dots, n_m , returns the code of the outcome of $\phi[\mathbf{x}/\bar{\mathbf{n}}]$, i.e., the code of $\phi(\bar{n}_0, \dots, \bar{n}_m)$. We will often write such a term as $\phi(\dot{\mathbf{x}})$.

Note that we may also use this notation in the meta-language. As is standard, we may define $X \subseteq Y$ by $\forall x(x \in X \rightarrow x \in Y)$, and $X = Y$ by $X \subseteq Y \wedge Y \subseteq X$. If the set F is meant to represent a function, we may write $y = F(x)$ instead of $\langle x, y \rangle \in F$. *Sequents* will be first-order objects of the form $\gamma = \langle \gamma_1, \dots, \gamma_n \rangle$, where each γ_i is a formula. We will treat sequents as sets, defining $\phi \in \gamma$ by $\exists i < |\gamma| \phi = (\gamma)_i$, and define $\gamma \subseteq \delta$ similarly. The difference between the first- and second-order use of these symbols will be clarified by the use of uppercase or lowercase letters. We may write γ, ϕ or (γ, ϕ) instead of $\gamma \frown \phi$. We similarly use γ, δ to denote the concatenation of γ and δ . The empty sequent will be denoted by \perp ; observe that we do not take it to be a symbol of our formal language.

2.2. Basic rules and axioms

We will work with a one-sided Tait-style calculus, which proves sequents of the form $\gamma = \langle \gamma_i \rangle_{i < n}$, as defined in e.g. [15]. In such a calculus, negation may only be applied to atomic formulas. We assume that the Tait calculus is formalized in such a way that the scheme stating that γ, α is derivable whenever α is a true atomic sentence is provable in ECA_0 ; this is not a strong assumption, as Σ_1^0 -completeness is provable in EA for standard calculi [10].

We will also assume that at least the following rules are available:

$$\begin{array}{ll}
(\text{LEM}) & \frac{}{\gamma, \alpha, \sim\alpha} & (=) & \frac{\gamma, \alpha \quad \gamma, r = r'}{\gamma, \alpha'} \\
(\wedge) & \frac{\gamma, \phi \quad \gamma, \psi}{\gamma, \phi \wedge \psi} & (\vee) & \frac{\gamma, \phi, \psi}{\gamma, \phi \vee \psi} \\
(\forall^0) & \frac{\gamma, \phi(v)}{\gamma, \forall x \phi(x)} & (\exists^0) & \frac{\gamma, \phi(t)}{\gamma, \exists x \phi(x)} \\
(\forall^1) & \frac{\gamma, \phi(V)}{\gamma, \forall X \phi(X)} & (\exists^1) & \frac{\gamma, \phi(Y)}{\gamma, \exists X \phi(X)} \\
(\text{CUT}) & \frac{\gamma, \phi \quad \gamma, \sim\phi}{\gamma}, & &
\end{array}$$

where α is atomic, v, V do not appear free in γ , and α' is obtained from α by replacing some instances of r by r' . We denote this calculus by TAIT ; TAIT^ρ is the restriction of TAIT which allows cuts only for formulas of rank less than $\rho \leq \omega$ (in particular, $\text{TAIT} = \text{TAIT}^\omega$).

2.3. Successor induction and comprehension

As our ‘background theory’ we will use Robinson’s arithmetic \mathbf{Q} [10] (essentially, PA without induction), enriched with axioms for the exponential; call the resulting theory \mathbf{Q}^+ . Aside from the basic axioms of \mathbf{Q}^+ , the following schemes will be useful in axiomatizing many theories of interest to us. Below, Γ denotes a set of formulas.

Γ -CA: $\exists X \forall x (x \in X \leftrightarrow \phi(x))$, where $\phi \in \Gamma$ and X is not free in ϕ ;

Δ_1^0 -CA: $\forall x (\pi(x) \leftrightarrow \sigma(x)) \rightarrow \exists X \forall x (x \in X \leftrightarrow \sigma(x))$,
where $\sigma \in \Sigma_1^0$, $\pi \in \Pi_1^0$, and X is not free in σ or π ;

IF : $\phi(0) \wedge \forall x (\phi(x) \rightarrow \phi(x+1)) \rightarrow \forall x \phi(x)$, where $\phi \in \Gamma$;

Ind: $0 \in X \wedge \forall x (x \in X \rightarrow x+1 \in X) \rightarrow \forall x (x \in X)$.

With this, we may define the following theories:

$$\begin{aligned}
\text{ECA}_0 &\equiv \text{Q}^+ + \text{Ind} + \Delta_0^0\text{-CA}; \\
\text{RCA}_0^* &\equiv \text{Q}^+ + \text{Ind} + \Delta_1^0\text{-CA}; \\
\text{RCA}_0 &\equiv \text{Q}^+ + \text{I}\Sigma_1^0 + \Delta_1^0\text{-CA}; \\
\text{ACA}_0 &\equiv \text{Q}^+ + \text{Ind} + \Sigma_1^0\text{-CA}; \\
\Pi_1^1\text{-CA}_0 &\equiv \text{Q}^+ + \text{Ind} + \Pi_1^1\text{-CA}.
\end{aligned}$$

Recall that we have included the exponential as a function symbol in our language; without it, RCA_0^* would require an additional axiom **Exp** stating that the exponential is total. In the case of ECA_0 , an alternative presentation without an exponential symbol would be less natural. Later we will make use of the fact that (in particular) ACA_0 is finitely axiomatizable [17, Lemma VIII.1.5].

Next, it will be useful to give a somewhat more economical (but equivalent) representation of $\Pi_1^1\text{-CA}_0$.

Theorem 2.1. *The theory $\Pi_1^1\text{-CA}_0$ is equivalent to*

$$\text{Q}^+ + \text{Ind} + (\Pi_1^1/\Sigma_2^0)\text{-CA}.$$

Proof sketch. In [17, Lemma V.1.4], it is proven that any Π_1^1 formula is equivalent to one of the form $\forall f: \mathbb{N} \rightarrow \mathbb{N} \phi(f)$, where $\phi \in \Sigma_1^0$. If $\text{fun}(F) \in \Pi_2^0(F)$ is a formula stating that F is the graph of a function, this is in turn equivalent to some formula $\forall F (\sim \text{fun}(F) \vee \phi'(F)) \in \Pi_1^1/\Sigma_2^0$, where ϕ' is obtained by modifying ϕ in the obvious way. \square

2.4. Transfinite recursion and bar induction

We mention two further theories that will appear later and require a more elaborate setup. We may represent well-orders in second-order arithmetic as pairs of sets $\Lambda = \langle |\Lambda|, <_\Lambda \rangle$, and define

$$\begin{aligned}
\text{Prog}_\phi(\Lambda) &= \forall \lambda \left((\forall \xi <_\Lambda \lambda \phi(\xi)) \rightarrow \phi(\lambda) \right) \\
\text{TI}_\phi(\Lambda) &= \forall \lambda \in |\Lambda| \left(\text{Prog}_\phi(\Lambda) \rightarrow \phi(\lambda) \right) \\
\text{WF}(\Lambda) &= \forall X \text{ TI}_{\lambda \in X}(\Lambda) \\
\text{WO}(\Lambda) &= \text{LO}(\Lambda) \wedge \text{WF}(\Lambda),
\end{aligned}$$

where $\text{LO}(\Lambda)$ is a formula expressing that Λ is a linear order.

Given a set X whose elements we will regard as ordered pairs $\langle \lambda, n \rangle$, let X_λ be the set of all n with $\langle \lambda, n \rangle \in X$, and $X_{<_\Lambda \lambda}$ be the set of all $\langle \eta, n \rangle$ with $\eta <_\Lambda \lambda$. With this, we define the *transfinite recursion* scheme by

$$\mathbf{TR}_\phi(X, \Lambda) = \forall \lambda \in |\Lambda| \forall n (n \in X_\lambda \leftrightarrow \phi(n, X_{<_\Lambda \lambda})).$$

Finally, we define

$$\begin{aligned} \mathbf{ATR}_0 &\equiv \mathbf{ACA}_0 + \left\{ \forall \Lambda (\mathbf{WO}(\Lambda) \rightarrow \exists X \mathbf{TR}_\phi(X, \Lambda)) : \phi \in \mathbf{\Pi}_\omega^0 \right\}; \\ \Gamma\text{-}\mathbf{BI}_0 &\equiv \mathbf{ACA}_0 + \left\{ \forall \Lambda (\mathbf{WO}(\Lambda) \rightarrow \mathbf{TI}_\phi(\Lambda)) : \phi \in \Gamma \right\}. \end{aligned}$$

These theories are rather powerful, yet as we will see, $\mathbf{\Pi}_1^1\text{-CA}_0$ proves very strong reflection principles for both of them; this is particularly remarkable in the case of $\mathbf{\Pi}_\omega^1\text{-BI}_0$, which is not a subtheory of $\mathbf{\Pi}_1^1\text{-CA}_0$. The following is proven in [16]:

Lemma 2.2. $\mathbf{\Pi}_1^1\text{-BI}_0 \not\subseteq \mathbf{ATR}_0 \subsetneq \mathbf{\Sigma}_1^1\text{-BI}_0$.

To be precise, $\mathbf{\Pi}_1^1\text{-BI}_0 \equiv \mathbf{\Sigma}_1^1\text{-DC}_0$, a theory known to be incomparable with \mathbf{ATR}_0 .

3. Formalized ω -logic

In this section we will give the necessary definitions in order to reason about ω -logic within second-order arithmetic, and introduce the provability operator $[P]$ based on ω -proofs.

3.1. Formalized deduction

For our purposes, a *theory* is a set of sequents defined by an arithmetical formula $\Box_T \gamma$, where γ is a first-order variable. For $\rho \leq \omega$, fix $\mathbf{Rule}^\rho(x, y) \in \Delta_0^0$ such that it is provable in \mathbf{ECA}_0 that if $\mathbf{Rule}^\rho(x, y)$ holds, then x codes a sequence of sequents $\langle \delta_i \rangle_{i < n}$ and y codes a sequent γ , and such that $\frac{\langle \delta_i \rangle_{i < n}}{\gamma}$ is an instance of a rule of \mathbf{TAIT}^ρ if and only if $\mathbf{Rule}^\rho(\langle \delta_i \rangle_{i < n}, \gamma)$ holds.

We also need to formalize the infinitary Tait calculus with the ω -rule, which we denote by $\omega\text{-TAIT}$. Recall that this rule has infinitely many premises, and has the following form:

$$\frac{\langle \gamma, \phi(\bar{n}) : n \in \omega \rangle}{\gamma, \forall x \phi(x)}.$$

We can formalize this using the following expression:

$$\omega\text{-Rule}(P, \gamma) \equiv \exists \phi \in \gamma \exists x, \psi < \phi \left(\phi = \forall x \psi(x) \wedge \forall z (\gamma, \psi(\dot{z}) \in P) \right).$$

Here, P is a set-variable. The formula $\omega\text{-Rule}(P, \gamma)$ states that γ follows by applying one ω -rule to elements of P , and will be used in our formalizations of ω -logic.

3.2. Theories with oracles

In order to deal with free second-order variables, we will enrich theories with oracles. As we have mentioned previously, we will use countably many constants $\mathbf{O} = \langle O_i \rangle_{i \in \mathbb{N}}$ in order to ‘feed’ information about any tuple of sets of numbers into T . The O_i ’s are assumed to be disjoint from the second-order variables.

To be precise, we first encode finite sequences of sets in a natural way: for example, we may encode $\langle A_i \rangle_{i < n}$ by

$$\mathbf{A} = \{ \langle 0, n \rangle \} \cup \{ \langle k, i + 1 \rangle : k \in A_i \wedge i < n \}.$$

The pair $\langle 0, n \rangle$ is included in order to know the length of the sequence, in case that e.g. $A_{n-1} = \emptyset$. As with tuples of natural numbers, let us write $n = |\mathbf{A}|$.

Then, given a Tait theory T and a set-tuple \mathbf{A} , define $T|\mathbf{A}$ to be the theory whose rules and axioms are those of T together with the new rules

$$(\text{O}_\in) \quad \frac{}{\gamma, \bar{k} \in O_i} \quad \text{for } k \in A_i \text{ and } i < |\mathbf{A}|$$

$$(\text{O}_\notin) \quad \frac{}{\gamma, \bar{k} \notin O_i} \quad \text{for } k \notin A_i \text{ and } i < |\mathbf{A}|.$$

It should be clear that these rules can be defined by some arithmetical formula $\text{OrAx}(y, \mathbf{A})$ and we define $\text{Rule}_{T|\mathbf{A}}(x, y) = \text{Rule}(x, y) \vee \text{OrAx}_T(x, \mathbf{A})$. If T is a Tait theory, we will say $T|\mathbf{A}$ is a *Tait theory with oracles*. When working in $T|A_1, \dots, A_n$ we may write $x \in \bar{A}_i$ instead of $x \in O_i$ to increase legibility; for example, instead of $\Box_{T|A,B}\phi(O_0, O_1)$, we may write $\Box_{T|A,B}\phi(\bar{A}, \bar{B})$.

3.3. Formalizing ω -logic using proof trees

In [1, 9], derivability in ω -logic is formalized by the existence of an (infinite) derivation tree. It will be convenient to use a standardized representation of such trees. Let $\mathbb{N}^{<\omega}$ denote the set of all finite sequences of natural numbers. We will represent ω -trees as subsets of $\mathbb{N}^{<\omega}$. If $\mathbf{s}, \mathbf{t} \in \mathbb{N}^{<\omega}$, define $\mathbf{s} \preceq \mathbf{t}$ if \mathbf{s} is an initial segment of \mathbf{t} , and $\downarrow \mathbf{s} = \{\mathbf{t} \in S : \mathbf{t} \preceq \mathbf{s}\}$. Then, say that an ω -tree is a set $S \subseteq \mathbb{N}^{<\omega}$ such that $\downarrow S = S$. A *labeled ω -tree* is a pair $\langle S, L \rangle$ such that S is an ω -tree and $L: S \rightarrow \mathbb{N}$.

Definition 3.1. A preproof (for T) of cut-rank at most $\rho \leq \omega$ is a labeled ω -tree $\langle S, L \rangle$ such that for every $\mathbf{s} \in S$, $L(\mathbf{s})$ is a sequent, and there is an instance $\frac{\langle \delta_i \rangle_{i < \xi}}{\gamma}$ of a rule of ω -TAIT $^\rho$ with $\xi \leq \omega$ such that $L(\mathbf{s}) = \gamma$ and for all $i \in \mathbb{N}$, $\mathbf{s} \frown i \in S$ if and only if $i < \xi$, in which case $L(\mathbf{s} \frown i) = \delta_i$, or else \mathbf{s} is a leaf and $T \vdash L(\mathbf{s})$. Let $\text{PreProof}_T^\rho(S, L)$ be a $\Pi_1^0(S, L)$ formula stating that $\langle S, L \rangle$ is a preproof for T of cut-rank at most ρ .

If S is (upwards) well-founded, we will say that $\langle S, L \rangle$ is an ω -proof.

The formula $\text{PreProof}_T^\rho(S, L)$ would make use of the formulas **Rule** and (a mild variant of) ω -**Rule** defined in §3.1; this is developed in much more detail, for example, in [9].

Definition 3.2. Given $\rho \leq \omega$, define a formula $[\mathbf{P}]_T^\rho \gamma$ by

$$\exists S \exists L \left(\text{WF}(\langle S, \succ \rangle) \wedge \text{PreProof}_T^\rho(S, L) \wedge L(\langle \rangle) = \gamma \right).$$

We write $[\mathbf{P}|X]_T^\rho \gamma$ instead of $[\mathbf{P}]_{T|X}^\rho \gamma$.

The following is immediate from the definition:

Lemma 3.3. Given $\rho \leq \sigma \leq \omega$, it is provable in ECA_0 that $[\mathbf{P}|X]_T^\rho \gamma$ implies $[\mathbf{P}|X]_T^\sigma \gamma$.

The notion of provability $[\mathbf{P}]$ gives rise to a natural reflection scheme.

Definition 3.4. Given a theory T , $\rho \leq \omega$, and a set of formulas Γ , we define a schema

$$\omega_{\mathbf{P}}\text{-RFN}_\Gamma^\rho[T] \equiv \forall \mathbf{A} \forall \mathbf{n} \left([\mathbf{P}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O}) \rightarrow \phi(\mathbf{n}, \mathbf{A}) \right),$$

where $\phi(\mathbf{z}, \mathbf{X}) \in \Gamma$ with all free variables shown.

We may omit the parameter ρ when $\rho = \omega$, as well as the parameter T when T is just the Tait calculus. This form of reflection gives an alternative axiomatization for bar induction, as shown by Arai [1].

Theorem 3.5. $\text{RCA}_0 + \omega_P\text{-RFN}_{\Pi^1_\omega} \equiv \text{RCA}_0 + \Pi^1_\omega\text{-BI}$.

Note the analogy with Kreisel and Lévy's result; just as reflection is equivalent to induction, ω -reflection is equivalent to transfinite induction. As we will see, different formulations of ω -logic can also give rise to certain forms of comprehension.

4. Countable ω -models and reflection

Another notion of reflection can be defined using ω -models. An ω -model is a second-order model whose first-order part consists of the standard natural numbers with the usual arithmetical operations. Because this part of our model is fixed, we only need to specify the second-order part, which consists of a family of sets over which we interpret second-order quantifiers. Moreover, if this family is countable, we can represent it using a *single* set.

In order to have names for all the sets appearing in our ω -model, we introduce countably many set-constants $\mathbf{C} = \langle C_i \rangle_{i < \omega}$ and let $\Pi^1_\omega(\mathbf{C})$ be the second-order language enriched with these constants. With this, a satisfaction notion can be associated to each countable coded ω -model in a natural way. If \mathbf{M} codes a sequence of sets, a satisfaction class on \mathbf{M} is a set which obeys the usual recursive clauses of Tarski's truth definition, where each constant C_n is interpreted as \mathbf{M}_n . Let us give a precise definition:

Definition 4.1. *Let $\mathbf{M} \subseteq \mathbb{N}$. A satisfaction class on \mathbf{M} is a set $S \subseteq \Pi^1_\omega(\mathbf{C})$ such that, for any terms t, s , $n \in \mathbb{N}$, and sentences ϕ, ψ ,*

$$\begin{aligned} (t \circ s) \in S &\Rightarrow \llbracket t \rrbracket \circ \llbracket s \rrbracket \quad (\circ \in \{=, \neq\}); \\ (t \circ C_n) \in S &\Rightarrow \langle n, \llbracket t \rrbracket \rangle \circ \mathbf{M} \quad (\circ \in \{\in, \notin\}); \\ (\phi \wedge \psi) \in S &\Rightarrow \phi \in S \text{ and } \psi \in S; \\ (\phi \vee \psi) \in S &\Rightarrow \phi \in S \text{ or } \psi \in S; \\ (\exists u \phi(u)) \in S &\Rightarrow \text{for some } n \in \mathbb{N}, \phi(\bar{n}) \in S; \\ (\forall u \phi(u)) \in S &\Rightarrow \text{for all } n \in \mathbb{N}, \phi(\bar{n}) \in S; \\ (\exists X \phi(X)) \in S &\Rightarrow \text{for some } n \in \mathbb{N}, \phi(C_n) \in S; \\ (\forall X \phi(X)) \in S &\Rightarrow \text{for all } n \in \mathbb{N}, \phi(C_n) \in S. \end{aligned}$$

Given a set of sentences $\Gamma \subseteq \Pi^1_\omega(\mathbf{C})$ closed under subformulas and substitution by closed terms (including set-constants), if for every $\phi \in \Gamma$ we have

that either $\phi \in S$ or $\sim\phi \in S$, we will say that S is a Γ -satisfaction class. If Γ contains all formulas of rank $\rho \leq \omega$, we say that S is a satisfaction class of rank ρ . A pair $\mathfrak{M} = \langle |\mathfrak{M}|, S_{\mathfrak{M}} \rangle$, where $|\mathfrak{M}|$ is a set and $S_{\mathfrak{M}}$ is a Γ -satisfaction class on $|\mathfrak{M}|$ of rank ρ is a Γ -valued ω -model of rank ρ . If Γ is the set of all sentences of $\Pi_{\omega}^1(\mathbf{C})$, we say that \mathfrak{M} is a full ω -model.

Satisfaction classes are used to define truth in a model:

Definition 4.2. Given an ω -model \mathfrak{M} , we write $\mathfrak{M} \models \phi$ if $\phi \in S_{\mathfrak{M}}$. If T is a theory, we say that \mathfrak{M} is a (partial) ω -model of T if, whenever ϕ is a theorem of T , it follows that $\mathfrak{M} \models \phi$. If \mathbf{A} is an a -tuple of sets, we write $[\mathbf{M}|\mathbf{A}]_T^{\rho}\phi$ for the formula stating that, for every Γ -valued ω -model \mathfrak{M} of rank at least ρ of T with $\phi \in \Gamma$ and $|\mathfrak{M}|_{<a} = \mathbf{A}$, $\mathfrak{M} \models \phi$.

Since the first-order part of an ω -model is just the natural numbers, it is easy to see that, for arithmetical sentences, truth in a model is equivalent to truth. This partially extends to Π_1^1 -sentences:

Lemma 4.3. Let T be any theory and $\rho \leq \omega$. Then, if $\phi(\mathbf{z}, \mathbf{X}) \in \Pi_1^1$ with all free variables shown,

$$\text{ECA}_0 \vdash \forall \mathbf{A} \forall \mathbf{n} (\phi(\mathbf{n}, \mathbf{A}) \rightarrow [\mathbf{M}|\mathbf{A}]_T^{\rho}\phi(\dot{\mathbf{n}}, \mathbf{C})).$$

Proof. First assume that ϕ is arithmetical, and let \mathfrak{M} be a model of T of rank ρ . Then, an external induction using the definition of a satisfaction class shows that, if ϕ holds, then $\mathfrak{M} \models \phi$. Otherwise, assume that $\phi = \forall X \psi(X)$ and $\mathfrak{M} \not\models \forall X \psi(X)$, so that $\mathfrak{M} \not\models \psi(C_k)$ for some k . But then, by the arithmetical case, $\psi(C_k)$ fails, so that $\forall X \psi(X)$ fails. \square

The following claim is immediate from observing that every model of rank σ is already a model of any rank $\rho \leq \sigma$:

Lemma 4.4. Let ϕ be an arbitrary formula and $\rho \leq \sigma \leq \omega$. Then,

$$\text{ECA}_0 \vdash \forall \mathbf{A} ([\mathbf{M}|\mathbf{A}]_T^{\rho}\phi(\mathbf{C}) \rightarrow [\mathbf{M}|\mathbf{A}]_T^{\sigma}\phi(\mathbf{C})).$$

We may use ω -models to define a notion of reflection $\omega_{\mathbf{M}}\text{-RFN}_{\Gamma}^{\rho}[T]$, analogously to Definition 3.4. The following is proven by Jäger and Strahm [11], and is a refinement of results of Friedman [8] and Simpson [16]:

Theorem 4.5. *Let $0 < n \leq \omega$, and fix a finite axiomatization of ACA_0 of rank ρ . Then,*

$$\text{ACA}_0 + \omega_{\text{M-RFN}}^{\rho}_{\Sigma_{1+n}^1} [\text{ACA}_0] \equiv \Pi_n^1\text{-BI}_0.$$

In fact, $[\text{P}]\gamma$ and $[\text{M}]\gamma$ are equivalent [9]. In the next section we will use inductive definitions to define two further notions of provability, which are also equivalent over a strong enough base theory.

Remark 4.6. *In the literature, ω -model reflection is often presented as ‘If ϕ is true, then ϕ is satisfiable in an ω -model’. We have presented it dually as ‘If ϕ holds in every ω -model, then ϕ is true’. The two schemes are clearly equivalent, but we prefer the latter for its symmetry with the other notions of reflection we consider. Note, however, that we must replace ϕ by $\sim\phi$ to pass from one to the other, and thus Theorem 4.5 is stated with Σ_{n+1}^1 in place of Π_{n+1}^1 as in [11].*

5. Inductive definitions of ω -logic

We may also formalize ‘provable in ω -logic’ in second-order arithmetic using a least fixed point construction. To this end, let us review how such fixed points may be treated in this framework.

5.1. Inductive definitions

Let us quickly review inductive definitions in the context of second-order arithmetic. Below, recall that we are working in a language without negation for non-atomic formulas.

Definition 5.1. *Let ϕ be any formula and X a set-variable. We say ϕ is positive on X if ϕ contains no occurrences of $t \notin X$.*

A positive formula ϕ induces a map $F = F_\phi: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$, which is monotone in the sense that $X \subseteq Y$ implies that $F(X) \subseteq F(Y)$. It is well-known that any such operator has a least fixed point.

Definition 5.2. *Given a formula $\phi(n, X)$, we define the abbreviations*

$$\begin{aligned} \text{Closed}_\phi(X) &\equiv \forall n (\phi(n, X) \rightarrow n \in X) \\ (X = \mu X.\phi) &\equiv \text{Closed}_\phi(X) \wedge \forall Y (\text{Closed}_\phi(Y) \rightarrow X \subseteq Y). \end{aligned}$$

It is readily checked that $n \in \mu X.\phi$ if and only if $\phi(n, \mu X.\phi)$ holds. Such fixed points can be constructed ‘from below’ using transfinite iterations of F : if we define $F^0(X) = X$, $F^{\xi+1}(X) = F(F^\xi(X))$ and $F^\xi(X) = \bigcup_{\zeta < \xi} F^\zeta(X)$, then by cardinality considerations one can see that

$$\mu X.\phi = F^{\omega_1}(\emptyset). \quad (6)$$

On the other hand, we may define $\mu X.\phi$ ‘from above’ as the intersection of all sets Y such that $\mathbf{Closed}(Y)$ holds. The latter definition is available in $\Pi_1^1\text{-CA}_0$, as is well-known (see e.g. [5]), and thus we see that:

Lemma 5.3. *Given $\phi(X) \in \Pi_\omega^0$ which is positive on X , it is provable in $\Pi_1^1\text{-CA}_0$ that $\exists Y (Y = \mu X.\phi)$.*

In particular, the rules of ω -logic give rise to a positive operator, and a theorem of ω -logic is any element of its least fixed point. Below, we develop this idea to give alternative formalizations of ω -logic.

5.2. The iterative formalization of ω -logic

We may use (6) to formalize ‘ ϕ is a theorem of ω -logic’, as in [6, 7]. There, provability along a countable well-order Λ is modeled using an ‘iterated provability class’ P , defined by arithmetical transfinite recursion as follows:

Definition 5.4. *Let Λ be a second-order variable that will be used to denote a well-order and T be a formal theory. Define $\mathbf{Iter}_T(\phi, P)$ to be the formula*

$$\Box_T \phi \vee \exists \psi (\omega\text{-Rule}(P, \psi) \wedge \Box_T(\psi \rightarrow \phi)).$$

Then, define

$$\begin{aligned} [\Lambda]_T \phi &\equiv \forall P (\mathbf{TR}_{\mathbf{Iter}_T}(P, \Lambda) \rightarrow \exists \lambda \in |\Lambda| (\phi \in P_\lambda)); \\ [\mathbf{R}]_T \phi &\equiv \exists \Lambda (\mathbf{WO}(\Lambda) \wedge [\Lambda]_T \phi). \end{aligned}$$

As before, write $[\mathbf{R}|\mathbf{A}]_T \phi$ instead of $[\mathbf{R}]_{T|\mathbf{A}} \phi$, and for a set of formulas Γ and $\rho \leq \omega$, define $\omega_{\mathbf{R}}\text{-RFN}_\Gamma^\rho[T]$ analogously to Definition 3.4.

Recall that, by our convention, the parameter ρ will be omitted when $\rho = \omega$. This form of reflection gives rise to an axiomatization of ATR_0 [6]:

Theorem 5.5. *Let U, T be c.e. theories such that $\text{ECA}_0 \subseteq U \subseteq \text{ATR}_0$, $\text{ECA}_0 \subseteq T$ and such that ATR_0 proves that any set X can be included in a full ω -model for T . Let Γ be any set of formulas such that $\{0 = 1\} \subseteq \Gamma \subseteq \Pi_2^1$. Then,*

$$\text{ATR}_0 \equiv U + \omega_R\text{-RFN}_\Gamma[T].$$

In Theorem 6.6, we will extend this result to reflection over higher complexity classes, and show that it also gives rise to an axiomatization of bar induction.

5.3. Formalizing ω -logic via a least fixed point

We obtain strictly more powerful reflection principles if we model ω -logic by an inductively defined fixed point, rather than its transfinite approximations.

Definition 5.6. *Fix a theory T , possibly with oracles, and $\rho \leq \omega$. Then, define a formula*

$$\text{SPC}_T^\rho(Q) \equiv Q = \mu P. \left(\Box_T \gamma \vee \exists \mathbf{x} \subseteq Q \text{Rule}^\rho(\mathbf{x}, \gamma) \vee \omega\text{-Rule}(Q, \gamma) \right).$$

If $\text{SPC}_T^\rho(Q)$ holds we will say that Q is a saturated provability class of rank ρ (ρ -SPC) for T .

With this, we may define our fixed point provability operator.

Definition 5.7. *We define a formula*

$$[\text{I}]_T^\rho \gamma \equiv \forall P \left(\text{SPC}_T^\rho(P) \rightarrow \gamma \in P \right).$$

We will write $[\text{I}|\mathbf{X}]_T^\rho \gamma$ instead of $[\text{I}]_{T|\mathbf{X}}^\rho \gamma$.

We will often want to apply this operator to formulas rather than sequents; when this is the case, we will identify a formula ϕ with the singleton sequent $\langle \phi \rangle$, and write $[\text{I}|\mathbf{X}]_T^\rho \phi$ instead of $[\text{I}|\mathbf{X}]_T^\rho \langle \phi \rangle$. Since SPC's are defined via an inductive definition, their existence can be readily proven in $\Pi_1^1\text{-CA}_0$.

Lemma 5.8. *Let T be any theory and $\rho \leq \omega$. Then, it is provable in $\Pi_1^1\text{-CA}_0$ that for every tuple of sets \mathbf{A} there exists a set P such that $\text{SPC}_{T|\mathbf{A}}^\rho(P)$ holds.*

Proof. Immediate from Lemma 5.3. □

It is important to note that we have defined $[|\mathbf{X}|_T^\rho \gamma$ by quantifying universally over all SPCs, so that $\sim[|\mathbf{X}|_T^\rho \gamma$ quantifies existentially over them. This means that such consistency statements automatically give us a bit of comprehension:

Lemma 5.9. *If T is any theory and γ any sequent, then*

$$\text{ECA}_0 \vdash \forall \mathbf{X} (\sim[|\mathbf{X}|_T^\rho \gamma \rightarrow \exists P \text{SPC}_{T|\mathbf{X}}^\rho(P)).$$

However, this instance of comprehension by itself does not necessarily carry additional consistency strength, in the following sense:

Lemma 5.10. *If T is a Tait theory extending ECA_0 ,*

$$T \equiv_{\Pi_1^0} T + \forall \mathbf{X} \exists P \text{SPC}_{T|\mathbf{X}}^\rho(P);$$

that is, the two theories prove the same Π_1^0 sentences.

This is proven in [7] for a weaker notion of provability, but the argument carries through in our setting. Roughly, we observe that $T + \Box_T \perp \equiv_{\Pi_1^0} T$, but $T + \Box_T \perp \vdash T + \forall \mathbf{X} \exists P \text{SPC}_{T|\mathbf{X}}^\rho(P)$, since in this case an SPC would simply consist of the set of all formulas.

Unlike the existence of SPCs, their *uniqueness* is immediate from their definition.

Lemma 5.11. *If T is any theory and $\rho \leq \omega$, we have that*

$$\text{ECA}_0 \vdash \forall \mathbf{X} \exists_{\leq 1} P \text{SPC}_{T|\mathbf{X}}^\rho(P),$$

where $\exists_{\leq 1} P \phi(P)$ is an abbreviation of $\forall P \forall Q (\phi(P) \wedge \phi(Q) \rightarrow P = Q)$.

As one might expect, adding new sets to our oracle gives us a stronger theory:

Lemma 5.12. *Let T be any theory and $\rho \leq \omega$. It is provable in ECA_0 that if \mathbf{A} is a tuple of sets and there exists an SPC for $T|\mathbf{A}$, then for any sequent γ and any set B ,*

$$[|\mathbf{A}|_T^\rho \gamma \rightarrow [|\mathbf{A}, B|_T^\rho \gamma.$$

Proof. Suppose that $[|\mathbf{A}|_T^\rho \gamma$. Using our assumption, we may choose an SPC P for $T|\mathbf{A}$, so that $\gamma \in P$. Let Q be an arbitrary SPC for $T|\mathbf{A}, B$. Observe that Q contains all axioms of $T|\mathbf{A}$ and is closed under all of its rules, so that by the minimality of P , we have that $P \subseteq Q$ and thus $\gamma \in Q$. Since Q was arbitrary, it follows that $[|\mathbf{A}, B|_T^\rho \gamma$, as needed. \square

Observe also that our least-fixed-point formalization of ω -provability is at least as strong as the formalization using ω -proofs:

Lemma 5.13. *Given any formula ϕ and $\rho \leq \sigma \leq \omega$, it is provable in ACA_0 that $[\mathbf{P}|\mathbf{A}]_T^\rho \gamma \rightarrow [\mathbf{I}|\mathbf{A}]_T^\sigma \gamma$.*

Proof. Assume that $[\mathbf{P}|\mathbf{A}]_T^\rho \gamma$ holds, and let $\langle S, L \rangle$ be an ω -proof of γ . Now, consider any SPC P , and consider the set $S' = \{\mathbf{s} \in S : L(\mathbf{s}) \notin P\}$, which is available in ACA_0 . By the closure conditions of P , one readily checks that S' cannot have a minimal element, and thus must be empty. In particular, $\gamma = L(\langle \rangle) \in P$. \square

Our goal now is to prove impredicative reflection within $\Pi_1^1\text{-CA}_0$. The following is a first approximation: $\Pi_1^1\text{-CA}_0$ proves that any formula proven in ω -logic with oracles is true in any ω -model.

Lemma 5.14 (ω -model soundness). *Given any theory T , a -tuple \mathbf{A} , and $\rho \leq \omega$,*

1. $\text{ACA}_0 \vdash \forall P \forall \mathbf{A} \forall \mathbf{n} \left(\text{SPC}_T^\rho(P) \wedge \ulcorner \phi(\dot{\mathbf{n}}, \mathbf{O}_{<a}) \urcorner \in P \rightarrow [\mathbf{M}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{C}_{<a}) \right);$
2. $\Pi_1^1\text{-CA}_0 \vdash \forall \mathbf{A} \forall \mathbf{n} \left([\mathbf{I}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O}_{<a}) \rightarrow [\mathbf{M}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{C}_{<a}) \right).$

Proof sketch. For the first claim, reason in ACA_0 . Let \mathfrak{M} be any model of T of rank ρ and let P be a saturated provability class for $T|\mathbf{A}$ of rank ρ . Let S' be obtained from $S_{\mathfrak{M}}$ by replacing each C_i with O_i if $i < a$ and by V_{a+i} otherwise. Then, S' is closed under all the rules and axioms defining P , so that, by minimality, $P \subseteq S'$. It follows that if $\phi(\mathbf{O}_{<a}) \in P$, then $\phi(\mathbf{O}_{<a}) \in P$ and so $\phi(\mathbf{C}_{<a}) \in S_{\mathfrak{M}}$; that is, $\mathfrak{M} \models \phi(\mathbf{C}_{<a})$.

The second claim then follows from the first, together with the provable existence of a unique ρ -SPC in $\Pi_1^1\text{-CA}_0$. \square

We remark that Lemma 5.14.1 may be formalized in a weaker theory, say RCA_0 . However, this will not be relevant for our main results.

6. Completeness and strong predicative reflection

In this section we will recall some completeness results for formalized ω -logic. It is well-known that ω -logic is Π_1^1 -complete [15], but it will be convenient to keep track of the second-order axioms needed to prove this. From these results, we will obtain a more general form of Theorem 5.5.

6.1. Completeness results for ω -logic

We begin with a weak completeness result available in ECA_0 .

Lemma 6.1. *Fix a theory T and $\rho \leq \omega$. Let $\gamma(\mathbf{z}, \mathbf{X}) \subseteq \Pi_\omega^0$ with all free variables shown. Then, it is provable in ECA_0 that*

$$\forall \mathbf{A} \forall \mathbf{n} \left(\bigvee \gamma(\mathbf{n}, \mathbf{A}) \rightarrow [\|\mathbf{A}\|_T^\rho \gamma(\dot{\mathbf{n}}, \mathbf{O}) \right). \quad (7)$$

Proof. Reasoning within ECA_0 , fix a tuple \mathbf{n} of natural numbers and \mathbf{A} of sets and assume that $\bigvee \gamma(\mathbf{n}, \mathbf{A})$ holds, and write $\gamma = (\delta, \phi)$ so that $\phi \in \gamma$ holds. We proceed by an external induction on ϕ . Assume that P is an arbitrary SPC for $T|\mathbf{X}$; we must prove that $(\delta, \phi(\bar{\mathbf{n}}, \mathbf{O})) \in P$. If ϕ does not contain quantifiers we proceed as in a standard Σ_1^0 -completeness proof, as in e.g. [10, pp. 175–176]; we omit the details, but remark that the case for atomic formulas requires a secondary external induction on the complexity of the terms that may appear.

Now assume that ϕ contains quantifiers. Let us consider the case where $\phi = \forall x \theta$. By the external induction hypothesis we have, for every k , that

$$(\delta, \theta(\bar{k}, \bar{\mathbf{n}}, \mathbf{O})) \in P.$$

But, P is closed under the ω -rule, so we also have that

$$(\delta, \forall x \theta(x, \bar{\mathbf{n}}, \mathbf{O})) \in P.$$

The remaining cases follow a similar structure; the case where ϕ is a Boolean combination of its subformulas is straightforward using the rules of the Tait calculus, and if $\phi = \exists x \theta(x)$, then for some k we have that $\theta(\bar{k})$ is true and we may use the induction hypothesis plus existential introduction. \square

So, ECA_0 already proves the completeness of ω -logic for arithmetical formulas, but we need to turn to ACA_0 to prove that it is also complete for Π_1^1 formulas. The following is a mild modification of the Henkin-Orey ω -completeness theorem [9, 14]:

Theorem 6.2. *For any formula $\phi(\mathbf{X}) \in \Pi_\omega^1$ and $\rho \leq \omega$,*

$$\text{ACA}_0 \vdash \forall \mathbf{A} \forall \mathbf{n} \left([\mathbf{M}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O}) \rightarrow [\mathbf{P}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{C}) \right).$$

The following is then immediate from Lemma 5.13:

Corollary 6.3. *For any formula $\phi(\mathbf{X}) \in \Pi_\omega^1$ and any $\rho \leq \omega$,*

$$\text{ACA}_0 \vdash \forall \mathbf{A} \forall \mathbf{n} \left([\mathbf{M}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O}) \rightarrow [\mathbf{I}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{C}) \right).$$

For formulas of relatively low complexity, we can replace $[\mathbf{M}|\mathbf{A}]_T^\rho \phi$ by ϕ :

Corollary 6.4. *Let $\rho \leq \omega$.*

1. *Given $\phi(\mathbf{z}, \mathbf{X}) \in \Pi_1^1$ with all free variables shown,*

$$\text{ACA}_0 \vdash \forall \mathbf{A} \forall \mathbf{n} \left(\phi(\mathbf{n}, \mathbf{A}) \rightarrow [\mathbf{I}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O}) \right).$$

2. *Given $\phi(\mathbf{z}, \mathbf{X}) \in \Sigma_2^1$ with all free variables shown,*

$$\text{ACA}_0 \vdash \forall \mathbf{A} \forall \mathbf{n} \left(\phi(\mathbf{n}, \mathbf{A}) \rightarrow \exists B [\mathbf{I}|\mathbf{A}, B]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O}) \right).$$

Proof. The first claim is immediate from Lemma 4.3 and Corollary 6.3. For the second, suppose that $\phi(\mathbf{z}, \mathbf{X}) = \exists Y \psi(\mathbf{z}, \mathbf{X}, Y)$, with $\psi \in \Pi_1^1(\mathbf{X}, Y)$. Then, if $\phi(\mathbf{n}, \mathbf{A})$ holds we can fix B so that $\psi(\mathbf{n}, \mathbf{A}, B)$ is the case, and we may use the first claim to conclude that $[\mathbf{I}|\mathbf{A}, B]_T^\rho \psi(\dot{\mathbf{n}}, \mathbf{O}, \bar{B})$, so that by existential introduction we have $[\mathbf{I}|\mathbf{A}, B]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O})$. \square

6.2. Predicative reflection and bar induction

Using the results we have discussed on completeness of ω -logic and Theorem 4.5, we may extend Theorem 5.5 to consider reflection for higher complexity classes. Below, recall that the parameter ρ may be omitted when $\rho = \omega$.

Lemma 6.5. *Let T be any theory. Then, over ATR_0 , the following are provably equivalent: 1. $[\mathbf{P}|\mathbf{A}]_T \phi$, 2. $[\mathbf{M}|\mathbf{A}]_T \phi$, 3. $[\mathbf{R}|\mathbf{A}]_T \phi$.*

Proof. That 3 implies 2 is proven in [6], and that 2 implies 1 follows from Theorem 6.2. Thus it remains to show that 1 implies 3.

Reasoning in ATR_0 , suppose that $\langle S, L \rangle$ is an ω -proof of ϕ . We use a well-known technique of ‘linearizing’ \preceq , as in e.g. [1]. Consider the ordering \trianglelefteq on S given by $\mathbf{s} \trianglelefteq \mathbf{t}$ if one of the following occurs: (a) $\mathbf{t} \preceq \mathbf{s}$, or (b) \mathbf{s}, \mathbf{t} are incomparable under \preceq , and for the least i such that $\mathbf{s}_i \neq \mathbf{t}_i$, we have that $(\mathbf{s})_i \leq (\mathbf{t})_i$. Then, it is readily verified that \trianglelefteq is a well-order on S . Using arithmetical transfinite recursion, let P be an IPC for $T|\mathbf{A}$ along $\langle S, \trianglelefteq \rangle$.

Then, a straightforward transfinite induction along \preceq shows that, for all $\mathbf{s} \in S$, $\bigvee L(\mathbf{s}) \in P_{\mathbf{s}}$; in particular, $\phi \in P_{\emptyset}$. Since P was arbitrary, we conclude that $[\mathbf{R}|\mathbf{A}]_T \phi$. \square

Theorem 6.6. *Let U be a theory such that $\text{ECA}_0 \subseteq U \subseteq \text{ATR}_0$. Then, for any $n \leq \omega$,*

$$\text{ATR}_0 + \Pi_n^1\text{-BI} \equiv U + \omega_{\mathbf{R}\text{-RFN}_{\Sigma_{1+n}^1}}[\text{ACA}_0]. \quad (8)$$

Proof. The case for $n = 0$ follows from Theorem 5.5, in view of the fact that $\text{ATR}_0 \vdash \Pi_0^1\text{-BI}$, so we assume $n > 0$. Let $R \equiv U + \omega_{\mathbf{R}\text{-RFN}_{\Sigma_{1+n}^1}}[\text{ACA}_0]$. Let ρ be the rank of an axiomatization of ACA_0 . Note that by Theorem 5.5, $\text{ATR}_0 \subseteq R$, and hence $R \equiv \text{ATR}_0 + \omega_{\mathbf{R}\text{-RFN}_{\Sigma_{1+n}^1}}[\text{ACA}_0]$. But, in view of Lemma 6.5,

$$R \equiv \text{ATR}_0 + \omega_{\mathbf{M}\text{-RFN}_{\Sigma_{1+n}^1}}[\text{ACA}_0] \equiv \text{ATR}_0 + \omega_{\mathbf{M}\text{-RFN}_{\Sigma_{1+n}^1}^\rho}[\text{ACA}_0],$$

where the second equivalence is due to the fact that ATR_0 proves that any satisfaction class extends to a full satisfaction class. But, by Theorem 4.5,

$$\text{ATR}_0 + \omega_{\mathbf{M}\text{-RFN}_{\Sigma_{1+n}^1}^\rho}[\text{ACA}_0] \equiv \text{ATR}_0 + \Pi_n^1\text{-BI},$$

as needed. \square

In view of Lemma 2.2, it follows that Theorem 5.5 is sharp:

Corollary 6.7. $\text{ATR}_0 \not\vdash \omega_{\mathbf{R}\text{-RFN}_{\Sigma_2^1}}[\text{ACA}_0]$.

Remark 6.8. *We could instead use Theorem 3.5 to obtain a variant of Theorem 6.6 with the pure Tait calculus in place of ACA_0 . For greater generality, it may be of interest to analyze the proof in [11] to identify the minimal requirements on a theory T which would allow us to replace ACA_0 by T .*

7. Consistency and reflection using inductive definitions

In this section we will define the notions of reflection and consistency that naturally correspond to $[\mathbf{I}|\mathbf{A}]_T^\rho$. Moreover, we will link the two notions to each other and see how they relate to comprehension. Below, recall that \perp denotes the empty sequent.

Definition 7.1. Given a theory T , $\rho \leq \omega$, and a class of formulas Γ , we define the schemas

$$\begin{aligned}\omega_1\text{-RFN}_\Gamma^\rho[T] &= \forall \mathbf{A} \forall \mathbf{n} \left([\mathbb{I}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O}) \rightarrow \phi(\mathbf{n}, \mathbf{A}) \right), \\ \omega_1\text{-CONS}_\Gamma^\rho[T] &= \forall \mathbf{A} \forall \mathbf{n} \sim \left([\mathbb{I}|\mathbf{A}]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O}) \wedge [\mathbb{I}|\mathbf{A}]_T^\rho \sim \phi(\dot{\mathbf{n}}, \mathbf{O}) \right), \\ \omega_1\text{-Cons}^\rho[T] &= \forall \mathbf{A} \sim [\mathbb{I}|\mathbf{A}]_T^\rho \perp,\end{aligned}$$

for $\phi(\mathbf{z}, \mathbf{X}) \in \Gamma$ with all free variables shown.

Lemma 7.2. Given any theory T ,

1. if $\rho \leq \omega$, $\text{ACA}_0 + \omega_1\text{-RFN}_\Gamma^\rho[T] \vdash \omega_M\text{-RFN}_\Gamma^\rho[T]$;
2. if $\rho \leq \omega$, $\Pi_1^1\text{-CA}_0 + \omega_1\text{-RFN}_\Gamma^\rho \equiv \Pi_1^1\text{-CA}_0 + \omega_M\text{-RFN}_\Gamma^\rho[T]$.

Proof. For the first claim, reason in $\text{ACA}_0 + \omega_1\text{-RFN}_\Gamma^\rho[T]$. Suppose that $\phi \in \Gamma$ and $[\mathbb{M}|\mathbf{A}]_T^\rho \phi(\bar{\mathbf{n}}, \mathbf{C})$. Then, by Corollary 6.3, $[\mathbb{I}|\mathbf{A}]_T^\rho \phi(\bar{\mathbf{n}}, \mathbf{O})$, and thus $\phi(\mathbf{n}, \mathbf{A})$ holds by $\omega_1\text{-RFN}_\Gamma^\rho$. For the second claim, the remaining inclusion follows from Lemma 5.14. \square

Of course, the schema $\omega_1\text{-CONS}_\Gamma^\rho[T]$ is only interesting when $\rho < \omega$, since otherwise it is just equivalent to consistency.

Lemma 7.3. If T is any theory and $\rho \leq \omega$, then

$$\text{ECA}_0 + \omega_1\text{-CONS}_{\Pi_\omega^1}^\rho[T] \subseteq \text{ECA}_0 + \omega_1\text{-Cons}^\omega[T].$$

Proof. Reasoning by contrapositive, if $\omega_1\text{-CONS}_{\Pi_\omega^1}^\rho[T]$ fails, then for some formula $\phi(\mathbf{z}, \mathbf{X})$, some tuple of sets \mathbf{A} and some tuple of natural numbers \mathbf{n} , we have that

$$[\mathbb{I}|\mathbf{A}]_T^\rho \phi(\bar{\mathbf{n}}, \mathbf{O}) \wedge [\mathbb{I}|\mathbf{A}]_T^\rho \sim \phi(\bar{\mathbf{n}}, \mathbf{O}),$$

which applying one cut gives us $[\mathbb{I}|\mathbf{A}]_T^\omega \perp$. \square

Let us now see that with just a little amount of reflection we get arithmetical comprehension. The first step is to build new sets out of our provability operators.

Lemma 7.4. Let T be any Tait theory, $\phi(\mathbf{z}, \mathbf{X})$ be any formula and $\rho \leq \omega$. Then,

$$\text{ECA}_0 \vdash \forall \mathbf{A} \exists W \forall n \left(n \in W \leftrightarrow [\mathbb{I}|\mathbf{A}]_T^\rho \phi(\dot{n}, \mathbf{O}) \right).$$

Proof. Reason within ECA_0 and pick a tuple of sets \mathbf{A} . Consider two cases; if there does not exist a ρ -SPC for $T|\mathbf{A}$, then we may set $W = \mathbb{N}$ and observe that $\forall n (n \in W \leftrightarrow [\|\mathbf{A}\|_T^\rho \phi(\bar{n}, \mathbf{O}))$ holds trivially by vacuity.

If such an SPC does exist, by Lemma 5.11 it is unique; call it P . Within ECA_0 we may form the set

$$W = \{n : \phi(\bar{n}, \mathbf{O}) \in P\}.$$

Then, if $n \in W$ is arbitrary we have by the uniqueness of P that $[\|\mathbf{A}\|_T^\rho \phi(\bar{n}, \mathbf{O})]$ holds. Conversely, if $[\|\mathbf{A}\|_T^\rho \phi(\bar{n}, \mathbf{O})]$ holds, then in particular $\phi(\bar{n}, \mathbf{O}) \in P$ holds and $n \in W$ by definition, so W has all desired properties.

Since \mathbf{A} was arbitrary, the claim follows. \square

Lemma 7.5. *Let T be any theory and $\rho \leq \omega$. Then,*

$$\text{ACA}_0 \subseteq \text{ECA}_0 + \omega_1\text{-RFN}_{\Sigma_1^0}^\rho[T].$$

Proof. Work in $\text{ECA}_0 + \omega_1\text{-RFN}_{\Sigma_1^0}^\rho[T]$. We only need to prove $\Sigma_1^0\text{-CA}$, that is,

$$\forall \mathbf{X} \exists Y \forall n (n \in Y \leftrightarrow \phi(n, \mathbf{X})),$$

where $\phi(n, \mathbf{X})$ can be any formula in $\Sigma_1^0(\mathbf{X})$.

Fix some tuple of sets \mathbf{A} . By Lemma 7.4, we can form the set

$$Z = \{n : [\|\mathbf{A}\|_T^\rho \phi(\bar{n}, \mathbf{O})]\}.$$

We claim that $\forall n (n \in Z \leftrightarrow \phi(n, \mathbf{A}))$ which finishes the proof. If $n \in Z$, then, by reflection, $\phi(n, \mathbf{A})$. On the other hand, if $\phi(n, \mathbf{A})$ we get by arithmetical completeness (Lemma 6.1) that $[\|\mathbf{A}\|_T^\rho \phi(\bar{n}, \mathbf{O})]$, so that $n \in Z$. \square

The above result along with the completeness theorems mentioned earlier may be used to prove that many theories defined using reflection and consistency are equivalent. Below, $\sim\Gamma = \{\sim\phi : \phi \in \Gamma\}$.

Lemma 7.6. *Let T be a theory extending Q^+ , and $\rho \leq \omega$. Then:*

1. *if $\Sigma_1^0 \subseteq \Gamma \subseteq \Pi_1^1$,*

$$\text{ECA}_0 + \omega_1\text{-CONS}_\Gamma^\rho[T] \equiv \text{ECA}_0 + \omega_1\text{-RFN}_{\Gamma \cup \sim\Gamma}^\rho[T];$$

$$2. \text{ECA}_0 + \omega_1\text{-Cons}^\omega[T] \equiv \text{ECA}_0 + \omega_1\text{-RFN}_{\Pi_2^1}^\omega[T].$$

Proof. For the first claim, let us begin by proving that

$$\text{ECA}_0 + \omega_1\text{-CONS}_\Gamma^\rho[T] \subseteq \text{ECA}_0 + \omega_1\text{-RFN}_{\Gamma \cup \sim \Gamma}^\rho[T].$$

Assume $\omega_1\text{-RFN}_{\Gamma \cup \sim \Gamma}^\rho[T]$ and let $\phi \in \Gamma$. Towards a contradiction, suppose that for some tuple of natural numbers \mathbf{n} and some tuple of sets \mathbf{A} ,

$$[\mathbb{I}|\mathbf{A}]_T^\rho \phi(\bar{\mathbf{n}}, \mathbf{O}) \wedge [\mathbb{I}|\mathbf{A}]_T^\rho \sim \phi(\bar{\mathbf{n}}, \mathbf{O}).$$

By reflection, this gives us $\phi(\mathbf{n}, \mathbf{A}) \wedge \sim \phi(\mathbf{n}, \mathbf{A})$, which is impossible. Since ϕ was arbitrary, the claim follows.

Next we prove that

$$\text{ECA}_0 + \omega_1\text{-CONS}_\Gamma^\rho[T] \supseteq \text{ECA}_0 + \omega_1\text{-RFN}_{\Gamma \cup \sim \Gamma}^\rho[T].$$

For this, fix $\phi \in \Gamma \cup \sim \Gamma$ and reason in $\text{ECA}_0 + \omega_1\text{-CONS}_\Gamma^\rho[T]$. We first consider the case where $\phi = \phi(\mathbf{z}, \mathbf{X})$ is arithmetical.

Let \mathbf{n} be a tuple of natural numbers and \mathbf{A} a tuple of sets such that $[\mathbb{I}|\mathbf{A}]_T^\rho \phi(\bar{\mathbf{n}}, \mathbf{O})$. If $\phi(\mathbf{n}, \mathbf{A})$ were false, by Lemma 6.4.1, we would also have that $[\mathbb{I}|\mathbf{A}]_T^\rho \sim \phi(\bar{\mathbf{n}}, \mathbf{O})$; but this contradicts $\omega_1\text{-CONS}_\Gamma^\rho[T]$. We conclude that $\phi(\mathbf{n}, \mathbf{A})$ holds, as desired.

Before considering the case where ϕ is not arithmetical, observe that since $\Sigma_1^0 \subseteq \Gamma$, it follows that

$$\text{ECA}_0 + \omega_1\text{-CONS}_\Gamma^\rho[T] \supseteq \text{ECA}_0 + \omega_1\text{-RFN}_{\Sigma_1^0}^\rho[T],$$

and by Lemma 7.5, we have that

$$\text{ACA}_0 \subseteq \text{ECA}_0 + \omega_1\text{-CONS}_\Gamma^\rho[T],$$

so we may now use arithmetical comprehension.

With this observation in mind, the argument will be very similar to the one before. Once again, suppose that $[\mathbb{I}|\mathbf{A}]_T^\rho \phi(\bar{\mathbf{n}}, \mathbf{O})$ for some tuples \mathbf{n}, \mathbf{A} . If $\phi(\mathbf{n}, \mathbf{A})$ were false, by Corollary 6.4.2, there would be B such that $[\mathbb{I}|\mathbf{A}, B]_T^\rho \sim \phi(\bar{\mathbf{n}}, \mathbf{O})$. By Lemma 5.9, $\text{ECA}_0 + \omega_1\text{-CONS}_\Gamma^\rho[T]$ implies that there exists a ρ -SPC for $T|\mathbf{A}$, and hence we may use Lemma 5.12 to see that

$$[\mathbb{I}|\mathbf{A}, B]_T^\rho \phi(\bar{\mathbf{n}}, \mathbf{O}) \wedge [\mathbb{I}|\mathbf{A}, B]_T^\rho \sim \phi(\bar{\mathbf{n}}, \mathbf{O}).$$

As before, this contradicts $\omega_1\text{-CONS}_\Gamma^\rho[T]$. We conclude that $\phi(\mathbf{n}, \mathbf{A})$ holds, as desired.

Now we prove the second claim. The right-to-left implication is obvious, so we focus on the other. Reason in $\text{ECA}_0 + \omega_1\text{-Cons}^\omega[T]$. By Lemma 7.3, this implies $\omega_1\text{-CONS}_{\Pi_1^\omega}^\omega[T]$, so that using Lemma 7.5, we may reason in ACA_0 .

Fix $\phi(\mathbf{z}, \mathbf{X}) \in \Pi_2^1$ and assume that $[\![\mathbf{A}]\!]_T^\omega \phi(\bar{\mathbf{n}}, \mathbf{O})$. If $\phi(\mathbf{n}, \mathbf{A})$ were false, then by Corollary 6.4, we would also have $[\![\mathbf{A}, B]\!]_T^\omega \sim \phi(\bar{\mathbf{n}}, \mathbf{O})$ for some set B , and using Lemma 5.12 as above,

$$[\![\mathbf{A}, B]\!]_T^\omega \phi(\bar{\mathbf{n}}, \mathbf{O}) \wedge [\![\mathbf{A}, B]\!]_T^\omega \sim \phi(\bar{\mathbf{n}}, \mathbf{O}).$$

But this contradicts $\omega_1\text{-CONS}_{\Pi_1^\omega}^\omega[T]$, and we conclude that $\phi(X)$ holds. \square

Next, we turn our attention to proving that reflection implies $\Pi_1^1\text{-CA}_0$. This fact will be an easy consequence of the following:

Lemma 7.7. *Let T be any theory, $\rho \leq \omega$, $\Gamma \subseteq \Pi_\omega^0(\mathbf{X})$, and $\phi(\mathbf{z}, \mathbf{X}) \in \Pi_1^1/\Gamma$. Then, it is provable in $\text{ACA}_0 + \omega_1\text{-RFN}_{\Pi_1^1/\Gamma}^\rho[T]$ that*

$$\forall \mathbf{A} \forall \mathbf{n} \left(\phi(\mathbf{n}, \mathbf{A}) \leftrightarrow [\![\mathbf{A}]\!]_T^\rho \phi(\dot{\mathbf{n}}, \mathbf{O}) \right).$$

Proof. Reason in $\text{ACA}_0 + \omega_1\text{-RFN}_{\Pi_1^1/\Gamma}^\rho[T]$ and let \mathbf{A} and \mathbf{n} be arbitrary. For the left-to-right direction we see that if $\phi(\mathbf{n}, \mathbf{A})$ holds, then by provable Π_1^1 -completeness (Corollary 6.4), $[\![\mathbf{A}]\!]_T^\rho \phi(\bar{\mathbf{n}}, \mathbf{O})$ holds as well. For the right-to-left direction, if $[\![\mathbf{A}]\!]_T^\rho \phi(\bar{\mathbf{n}}, \mathbf{O})$, by $\omega_1\text{-RFN}_{\Pi_1^1/\Gamma}^\rho[T]$, $\phi(\mathbf{n}, \mathbf{A})$ holds. \square

We can now finally combine all our previous results and formulate the main theorem of this section.

Theorem 7.8. *Given any theory T ,*

$$\text{ACA}_0 + \omega_1\text{-RFN}_{\Pi_1^1/\Sigma_2^0}^\rho[T] \vdash \Pi_1^1\text{-CA}_0.$$

Proof. Work in $\text{ACA}_0 + \omega_1\text{-RFN}_{\Pi_1^1/\Sigma_2^0}^\rho[T]$. By Theorem 2.1, we need only prove comprehension for arbitrary $\phi(n, \mathbf{X}) \in \Pi_1^1/\Sigma_2^0(\mathbf{X})$.

Fix a tuple of sets \mathbf{A} . By Lemma 7.4, there is a set W satisfying

$$\forall n \left(n \in W \leftrightarrow [\![\mathbf{A}]\!]_T^\rho \phi(\dot{n}, \mathbf{O}) \right).$$

But by Lemma 7.7, this is equivalent to

$$\forall n \left(n \in W \leftrightarrow \phi(n, \mathbf{A}) \right).$$

Since ϕ and \mathbf{A} were arbitrary, we obtain $\Pi_1^1\text{-CA}_0$, as desired. \square

Thus impredicative reflection implies impredicative comprehension, as claimed. Next we will prove the opposite implication, but for this we will first need to take a detour through β -models.

8. Countable β -models and impredicative reflection

Our goal in this section is to derive a converse of Theorem 7.8. The main tool for this task will be the notion of a *countable coded β -model*. In what follows we shall discuss the definition and basic existence results for such models.

Note that the converse of Lemma 4.3 is not always true for Π_1^1 -sentences, as we are not truly quantifying over *all* subsets of \mathbb{N} . Nevertheless, for special kinds of models it may actually be the case that $\mathfrak{M} \models \forall X \phi(X)$ implies that $\forall X \phi(X)$ when ϕ is arithmetical; such models are called β -models.

Below, recall that $\mathbf{V} = \langle V_i \rangle_{i \in \mathbb{N}}$ is assumed to be a sequence listing all second-order variables, and that $\mathbf{S}_{<a} = \langle S_i \rangle_{i < a}$ for any sequence \mathbf{S} .

Definition 8.1. *A countable coded ω -model \mathfrak{M} is a β -model if for every $\phi(\mathbf{z}, \mathbf{V}_{<a}) \in \Pi_1^1$ and every \mathbf{n} , $\phi(\mathbf{n}, |\mathfrak{M}|_{<a})$ holds if and only if $\mathfrak{M} \models \phi(\bar{\mathbf{n}}, \mathbf{C}_{<a})$.*

Thus, β -models reflect Π_1^1 formulas; however, with no additional assumptions, we can push this property a bit farther.

Lemma 8.2. *Fix a formula $\phi(\mathbf{z}, \mathbf{V}_{<a}) \in \Sigma_2^1$. It is provable in ACA_0 that, for all a -tuples \mathbf{A} and all \mathbf{n} , if \mathfrak{M} is a β -model with $|\mathfrak{M}|_{<a} = \mathbf{A}$ and such that $\mathfrak{M} \models \phi(\bar{\mathbf{n}}, \mathbf{C}_{<a})$, then $\phi(\mathbf{n}, \mathbf{A})$ holds.*

Proof. Write $\phi = \exists X \forall Y \psi(\mathbf{z}, \mathbf{V}_{<a}, X, Y)$ and suppose that \mathbf{A} is an a -tuple of sets and \mathfrak{M} a model with $|\mathfrak{M}|_{<a} = \mathbf{A}$. Then, if $\mathfrak{M} \models \phi(\bar{\mathbf{n}}, \mathbf{C}_{<a})$, it follows that for some m , $\mathfrak{M} \models \forall Z \psi(\mathbf{C}_{<a}, C_m, Z)$. But since by assumption \mathfrak{M} is a β -model, it follows that $\forall Z \psi(\mathbf{A}, |\mathfrak{M}|_m, Z)$ holds, hence so does $\phi = \exists X \forall Y \psi(\mathbf{A}, X, Y)$. \square

A good part of the theory of β -models may be formalized within $\Pi_1^1\text{-CA}_0$. Theorems 8.3 and 8.4 may be found in [17]. Recall that we defined the theories ATR_0 and $\Pi_\omega^1\text{-BI}_0$ in Section 2.3.

Theorem 8.3. *It is provable in ATR_0 that, for every countable coded β -model \mathfrak{M} , $\mathfrak{M} \models \Pi_\omega^1\text{-BI}_0$.*

We remark that Theorem 8.3 obviously holds if we replace $\Pi_\omega^1\text{-BI}_0$ by a weaker theory, such as ACA_0 , ATR_0 , or others we have mentioned earlier. However, $\Pi_1^1\text{-CA}_0$ is required to *construct* β -models:

Theorem 8.4. *It is provable in $\Pi_1^1\text{-CA}_0$ that for every a -tuple of sets \mathbf{A} there is a full β -model \mathfrak{M} such that $|\mathfrak{M}|_{<a} = \mathbf{A}$.*

With these results in mind, we can now easily prove that comprehension implies reflection.

Lemma 8.5. *Let U, T be theories such that U extends ACA_0 and $\rho \leq \omega$. If U proves that any a -tuple \mathbf{A} can be included in an ω -model satisfying T of rank ρ , then for any $\phi(\mathbf{z}, \mathbf{X}) \in \Pi_2^1$ with all free variables shown, U proves that*

$$\forall P \forall \mathbf{A} \forall n \left(\text{SPC}_T^\rho(P) \wedge (\ulcorner \phi(\dot{n}, \mathbf{O}) \urcorner \in P) \rightarrow \phi(\mathbf{n}, \mathbf{A}) \right). \quad (9)$$

If U proves that any a -tuple \mathbf{A} can be included in a β -model satisfying T of rank $\rho \leq \omega$, (9) holds for $\phi \in \Pi_3^1$.

Proof. For the first claim, let $\phi(\mathbf{z}, \mathbf{V}_{<a}) = \forall X \psi(\mathbf{z}, \mathbf{V}_{<a}, X)$, where $\psi \in \Sigma_1^1$ with all free variables shown, and reason in ACA_0 . Fix an a -tuple \mathbf{A} of sets, a tuple of natural numbers \mathbf{n} , and a ρ -SPC P , and assume that $\phi(\bar{\mathbf{n}}, \mathbf{O}_{<a}) \in P$. Let B be arbitrary and \mathfrak{M} be an ω -model satisfying T with $|\mathfrak{M}|_{<a+1} = \mathbf{A}, B$. Then, by Lemma 5.14.1, $\mathfrak{M} \models \psi(\bar{\mathbf{n}}, \mathbf{C}_{<a}, C_a)$, so that by Lemma 8.2, $\psi(\mathbf{n}, \mathbf{A}, B)$ holds. Since B was arbitrary, we conclude that $\phi(\mathbf{n}, \mathbf{A}) = \forall X \psi(\mathbf{n}, \mathbf{A}, X)$ holds. The second claim is similar, but we take $\psi \in \Sigma_2^1$ and use Lemma 8.2. \square

Using the fact that $\Pi_1^1\text{-CA}_0$ proves the existence of a ρ -SPC, we obtain the following:

Corollary 8.6. *If $\rho \leq \omega$ and $\Pi_1^1\text{-CA}_0$ proves that any a -tuple \mathbf{A} can be included in a β -model for T of rank ρ , then*

$$\Pi_1^1\text{-CA}_0 \vdash \omega_1\text{-RFN}_{\Pi_3^1}^\rho[T].$$

We may now summarize our results in our main theorem.

Theorem 8.7. *Let U, T be theories such that $\text{ECA}_0 \subseteq U \subseteq \Pi_1^1\text{-CA}_0$, and such that $\Pi_1^1\text{-CA}_0$ proves that any set-tuple \mathbf{A} can be included in a β -model*

for T . Let $\Pi_1^1/\Sigma_2^0 \subseteq \Gamma \subseteq \Pi_3^1$. Then, for any $\rho \leq \omega$,

$$\begin{aligned} \Pi_1^1\text{-CA}_0 &\equiv U + \omega_1\text{-RFN}_\Gamma^\rho[T] \\ &\equiv U + \omega_1\text{-CONS}_\Gamma^\rho[T] \equiv U + \omega_1\text{-Cons}^\omega[T]. \end{aligned} \quad (10)$$

Proof. All inclusions are immediate from Lemmas 7.5 and 7.6, Theorem 7.8 and Corollary 8.6. \square

Corollary 8.8. Let $\mathcal{G} = \{\text{ECA}_0, \text{RCA}_0^*, \text{RCA}_0, \text{ACA}_0, \text{ATR}_0\}$ and $\rho \leq \omega$. Choose $U \in \mathcal{G} \cup \{\Pi_1^1\text{-CA}_0\}$ and $T \in \mathcal{G} \cup \{\text{TAIT}, \Pi_\omega^1\text{-BI}_0\}$. Then, (10) holds for U and T .

In view of Theorem 4.5, we may extend these results to reflection over higher complexity classes.

Theorem 8.9. Let U be a theory such that $\text{ECA}_0 \subseteq U \subseteq \Pi_1^1\text{-CA}_0$, and let ρ be the rank of some finite axiomatization of ACA_0 . Then, for any $n < \omega$ and $\sigma \in [\rho, \omega]$,

$$\Pi_1^1\text{-CA}_0 + \Pi_n^1\text{-BI} \equiv U + \omega_1\text{-RFN}_{\Sigma_{1+n}^1}^\sigma[\text{ACA}_0]. \quad (11)$$

Proof. As in the proof of Theorem 6.6, the case for $n = 0$ is immediate from Theorem 8.7, so we assume $n > 0$. Let $B = \Pi_1^1\text{-CA}_0 + \Pi_n^1\text{-BI}$ and $R = U + \omega_1\text{-RFN}_{\Sigma_{1+n}^1}^\sigma[\text{ACA}_0]$. First we show that $B \subseteq R$. Since $U + \omega_1\text{-RFN}_{\Pi_1^1/\Sigma_2^0}^\sigma[\text{ACA}_0] \subseteq R$, we obtain $\Pi_1^1\text{-CA}_0 \subseteq R$. We have that

$$\Pi_1^1\text{-CA}_0 + \omega_1\text{-RFN}_{\Sigma_{1+n}^1}^\sigma[\text{ACA}_0] \vdash \omega_M\text{-RFN}_{\Sigma_{1+n}^1}^\sigma[\text{ACA}_0]$$

by Lemma 7.2. But, $\Pi_1^1\text{-CA}_0 + \omega_M\text{-RFN}_{\Sigma_{1+n}^1}^\sigma[\text{ACA}_0] \vdash \omega_M\text{-RFN}_{\Sigma_{1+n}^1}^\rho[\text{ACA}_0]$ by Lemma 4.4, and we obtain $R \vdash \Sigma_n^1\text{-BI}$ by Theorem 4.5.

Next we show that $R \subseteq B$. By Theorem 4.5 and the fact that $\Pi_1^1\text{-CA}_0$ proves that any valuation can be extended to a full valuation, we have that $B \vdash \omega_M\text{-RFN}_{\Sigma_{1+n}^1}^\sigma[\text{ACA}_0]$. But, by Lemma 7.2,

$$\Pi_1^1\text{-CA}_0 + \omega_M\text{-RFN}_{\Sigma_{1+n}^1}^\sigma[\text{ACA}_0] \vdash \omega_1\text{-RFN}_{\Sigma_{1+n}^1}^\sigma[\text{ACA}_0].$$

Since $U \subseteq \Pi_1^1\text{-CA}_0$ by assumption, the result follows. \square

Remark 8.10. Note that $\omega_1\text{-RFN}_\Gamma^\rho[T]$ is equivalent to the conjunction of the two following statements:

- (1) Every ρ -SPC contains only true formulas from Γ ,
- (2) there exists a ρ -SPC containing any tuple of parameters \mathbf{A} .

Thus it is tempting to conjecture that either (1) or (2) is sufficient to obtain $\Pi_1^1\text{-CA}_0$. But this is not the case. Observe that ACA_0 proves that RCA_0 has ω -models of any finite rank ρ [17, Lemma VII.2.2], hence by Lemma 8.5, it proves that any ρ -SPC for RCA_0 reflects Π_2^1 formulas, yet $\text{ACA}_0 \subsetneq \Pi_1^1\text{-CA}_0$. Similarly, ATR_0 proves that ACA_0 has full ω -models [17, Theorem VIII.1.13], so it proves that any ω -SPC for ACA_0 reflects Π_2^1 formulas. We conclude that (1) is not sufficient.

Meanwhile, by Lemma 5.10, $T = \text{ACA}_0 + \exists P \text{SPC}_{\text{ACA}_0}^\rho(P)$ is equiconsistent with ACA_0 , hence $T \subsetneq \Pi_1^1\text{-CA}_0$. It follows that (2) is not sufficient either.

On the other hand, the reader may verify, using Lemma 8.5, that

$$\Pi_1^1\text{-CA}_0 \equiv \text{ACA}_0 + \exists P \text{SPC}_{\text{RCA}_0}^0(P) \equiv \text{ATR}_0 + \exists P \text{SPC}_{\text{ACA}_0}^\omega(P).$$

9. Concluding remarks

We have shown that $\Pi_1^1\text{-CA}_0$ and its extensions with bar induction are equivalent, over a weak base theory, to a family of proof-theoretic reflection or consistency assertions formalized using least fixed points. This, together with work on reflection principles based on ω -proofs and iterated approximations to a least fixed point, shows that many important systems of reverse mathematics may be represented in terms of reflection principles for ω -logic.

This immediately raises the question of whether stronger theories may be represented in a similar fashion, as well as theories in the language of (say) set theory. Such an endeavour would most likely require working with infinitary rules much stronger than the ω -rule, and may be a fruitful line of future inquiry.

A second natural question is whether these results will lead to a Π_1^0 ordinal analysis of these theories, in the style of Beklemishev's analysis of PA [3]. While this goal is part of the motivation for the present work, it is clear that this would require many further advances, both in the proof theory of reflection principles and in the study of transfinite provability logic.

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